Matching with Quantity

David Delacrétaz [*]

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Extended Abstract.
Preliminary and Incomplete.

Abstract

There are instances of two-sided matching problems where one side may be matched to several units on the other side. An example is the assignment of children to particular days at day care centers. The core may be empty in this context and when a stable matching exists the deferred acceptance algorithm may still yield an unstable one. The algorithm proposed in this paper iteratively eliminates choices that cannot be part of any stable matching until all children can receive their top choice. It always finds the child-optimal stable matching if it exists and confines blocking pairs to small groups of agents otherwise.

Finding a place for their child in a day care center is essential for parents who want to reconcile career and family. A large amount of evidence suggests that early childhood programs are important in the development of children, for example Chetty et al. (2011) found that these programs have a long-term impact on future earnings. Yet, finding a place can prove a very difficult task for parents. In Australia, a report from the Productivity Commission (2014) documented the dissatisfaction of parents who have difficulties finding a place that fits their needs despite a large increase in the number of places offered and government funding.

Following in the footsteps of Gale and Shapley (1962), economists have applied matching theory to various markets such as school choice (Abdulkadiroglu and Sönmez 2003) and kidney exchange (Roth, Sönmez and Ünver 2005, 2007). The matching of children to day care, however, has remained decentralized around the world. The Australian Productivity Commission (2014) reported many parents' complaints about the current decentralized system. In particular, parents tend to take the first place that becomes available even if it does not completely fit their needs because they cannot afford to risk not placing their child at all.

The main difference between the matching of children with day care centers and students with schools is that children often attend day care part-time while school attendance is always one a full time basis. A single place at a day care center can therefore be split between two or more children attending on different days. This seemingly small twist on the well known school choice model has important consequences, in particular there may not exist any stable matching and the deferred acceptance may yield an unstable matching even when a stable one does exist.

Kennes, Monte and Tumennasan (2014) study the assignment of children to day care centers in a dynamic context, taking into account the fact that children may move from one center to another

[*]Department of Economics, Level 3, FBE Building, University of Melbourne, 111 Barry St, Victoria 3010, Australia. Email: ddelacretaz@unimelb.edu.au.

once they have secured a place. The authors do not consider the part-time feature of childcare and effectively build a dynamic extension of the school choice model framed in the child care context.

Another dynamic aspect of day care is that children may enter and leave the market at any point in time. The vast majority of places are however likely to be vacated once a year when children are old enough to start school. There is therefore a clear need for a static centralized matching program that would facilitate the assignment of children to day care centers once a year when most places become available.

Sönmez and Switzer (2013) designed a matching market for the American Army that has a similar flavor to the model studied in this paper. Cadets are matched to army branches for a certain number of years. The existence of this third dimension means that a same pair of cadet and branch can be matched over different terms. Similarly, the possibility to choose different days means that a child and a day care center can be matched in several ways. The two models differ in the fact that the term is designed to give cadets an incentive to commit for a longer period and has no impact on the capacity of each branch. Children and day care centers are in contrast matched over different quantities and the less each child takes, the more children can be accommodated.

The closest model to this paper is the couple problem of the National Resident Matching Program (NRMP). The aim of the program is to match medical graduates to hospitals for their residency program. Graduates can apply as singles or as couples, in which case they rank pairs of hospitals in order of preferences. Similarly to the model developed in this paper, a stable matching may not exist and the deferred acceptance algorithm may fail to find one when the core is nonempty.

A satisfying 'engineering' solution has been developed for this specific situation (Roth and Peranson 1997, Roth 2003). Singles are matched first using the deferred acceptance algorithm. Couples are then introduced one at a time, following a random order. Whenever a couple enters, blocking pairs are matched together until none remains. This method has been shown to produce a stable matching whenever one exists (Klaus and Klijn 2005), however if multiple ones exist the order in which couples are introduced as well as the order in which blocking pairs are matched together will determine which one is selected.

Kojima, Pathak and Roth (2013) showed that a stable matching always exists and the core becomes small in large markets. The NRMP is indeed a very large market and although the algorithm lacks clear theoretical properties it has worked very well for this specific market.

The approach of this paper is different. While the motivation originally comes from the matching of children to day care centers, the algorithm developed in this paper uses a theory-based approach and is designed to fit a general class of models. Agents may want different quantities of different objects, which come in finite numbers of units. Each agent has preferences over packages of objects and is given a priority for each marginal unit of each object. In the context of child care, each family wants specific days at specific day care centers and limited places are available for each day. In the context of a couple problem, each agent is either a single who wants a place in a hospital or a couple that wants two places, possibly at different hospitals.

The algorithm developed in this paper is based on a simple idea. Looking at the priority list of children at each center, a child with a high enough priority can be guaranteed a place there because she will get either this choice or one she prefers in any stable matching. If a child is guaranteed one of her choices, all her subsequent ones can be removed from her preference list. A child may also have a low enough priority so that a choice could not be part of any stable matching and can be removed.

The algorithm iteratively eliminates those choices that cannot be part of a stable matching. If a stable matching exists, it will get to a point where all top choices can be accommodated. Each child is then given her top choice and the resulting allocation is the child-optimal stable matching.

If no stable matching exists, the algorithm will identify one or more *blocking groups*, subeconomies for which no stable matching exists. Any child who is not in a blocking group will not be part of a blocking pair and no unit she holds in the final allocation will be envied by someone with a higher priority. Blocking pairs will necessarily arise within each blocking group but will be confined to it. The algorithm therefore finds the child-optimal stable matching whenever it exists and gets as close as possible to it otherwise.

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